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A complete expression for the propagator corresponding to a model quadratic action

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Abstract. In this paper we give a complete expression for the propagator corresponding to the motion of a series of non-local harmonic oscillators under the influence of an arbitrary driving force. The method of derivation is based on a direct solution of the corresponding classical equation of motion. The time and end-point parameters are kept completely general throughout.

1. Introduction

Path-integral methods have for a long time been known to provide a powerful non-perturbative treatment for a variety of physical problems. It is very often the case that the retarded propagator may be expressed in terms of a path integral with an action containing a non-local term of the form

$$\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\sigma f(\boldsymbol{r}(\tau)-\boldsymbol{r}(\sigma)).$$

Examples are to be found in Feynman's pioneering work on the polaron [1], as well as in disordered systems [2]. However, since it is usually the case that an exact solution for the propagator cannot be found, we must resort to variational methods. The standard technique is to simulate the exact non-local action term with a non-local trial harmonic term of the form

$$\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\sigma\, g(|\tau-\sigma|)(\boldsymbol{r}(\tau)-\boldsymbol{r}(\sigma))^2.$$

A number of variational parameters may be included in this expression in order to get the best possible result.

In this paper we present an explicit derivation of the propagator corresponding to the following non-local harmonic action:

$$S=\tfrac{1}{2}m\int_0^t \mathrm{d}\tau\, \dot{\boldsymbol{r}}(\tau)^2-\tfrac{1}{8}\sum_{i=1}^{n}\kappa_i\Omega_i\int_0^t \mathrm{d}\tau\int_0^t \mathrm{d}\sigma \frac{\cos\Omega_i(t/2-|\tau-\sigma|)}{\sin\frac{1}{2}\Omega_i t}(\boldsymbol{r}(\tau)-\boldsymbol{r}(\sigma))^2 + \int_0^t \mathrm{d}\tau\, \boldsymbol{F}(\tau)\cdot\boldsymbol{r}(\tau) \tag{1.1}$$

where $\kappa_i>0$ and the parameters Ω_i are distinct. The final expression for the propagator will depend on the time parameter t and the end-point parameters $\boldsymbol{r}(t)$ and $\boldsymbol{r}(0)$. We emphasize that these parameters are kept absolutely general throughout the derivation.

The meanings of the terms in the action are as follows. The first term is the kinetic energy of the particle of mass m. The second term consists of a sum over n non-local oscillators, each one being of the form used in Feynman's work on the polaron [1]. The parameters κ_i and Ω_i may be used as variational parameters in applications. We have also included a term due to an arbitrary driving force $\boldsymbol{F}$. The physical situation described by the action may be understood in terms of the particle of mass m interacting with n fictitious particles of mass $M_i = \kappa_i/\Omega_i^2$.

A corresponding expression for the propagator in the case of one oscillator ($n = 1$) has been given by Sa-yakanit [3]. The method of derivation involved the introduction of a fictitious particle. More recently, this result was rederived by Adamowski *et al* [4] from a formalism relating to an arbitrary non-local harmonic action. However the direct method of derivation we choose here, valid for any number of oscillators, does not involve either fictitious particles or the formalism of [4].

Our basic formalism is as follows. In terms of a path integral, the expression for the propagator is

$$K(\boldsymbol{r}(t), t; \boldsymbol{r}(0), 0) = \int_{\boldsymbol{r}(0)}^{\boldsymbol{r}(t)} \mathrm{D}[\boldsymbol{r}(\tau)] \exp\left[\frac{\mathrm{i}}{\hbar} S\right]. \tag{1.2}$$

The integration is over all paths starting at time zero from $\boldsymbol{r}(0)$ and ending at time t at $\boldsymbol{r}(t)$. Since the action S is a quadratic function of the coordinates, it can be shown [5] that

$$K(\boldsymbol{r}(t), t; \boldsymbol{r}(0), 0) = G(t) \exp\left[\frac{\mathrm{i}}{\hbar} S_{\mathrm{cl}}(\boldsymbol{r}(t), \boldsymbol{r}(0), t)\right]. \tag{1.3}$$

Here S_{cl} is the classical action obtained by substituting the solution of the classical equation of motion into equation (1.1). The prefactor $G(t)$ is given by

$$G(t) = K_{\boldsymbol{F}=0}(0, t; 0, 0) = \int_0^0 \mathrm{D}[\boldsymbol{r}(\tau)] \exp\left[\frac{\mathrm{i}}{\hbar} S_{\boldsymbol{F}=0}\right] \tag{1.4}$$

and is independent of the coordinates $\boldsymbol{r}(t)$ and $\boldsymbol{r}(0)$. An explicit expression for this prefactor may be derived from the classical solution.

2. The classical solution

For the action given by equation (1.1), the classical equation of motion is

$$m\ddot{\boldsymbol{r}}(\tau) + \tfrac{1}{2}\sum_{i=1}^{n} \kappa_i\Omega_i \int_0^t \mathrm{d}\sigma \frac{\cos\Omega_i(t/2 - |\tau-\sigma|)}{\sin\frac{1}{2}\Omega_i t}(\boldsymbol{r}(\tau) - \boldsymbol{r}(\sigma)) = \boldsymbol{F}(\tau). \tag{2.1}$$

In order to solve this equation we first define

$$\boldsymbol{q}_i(\tau) = \tfrac{1}{2}\Omega_i \int_0^t \mathrm{d}\sigma \frac{\cos\Omega_i(t/2 - |\tau-\sigma|)}{\sin\frac{1}{2}\Omega_i t} \boldsymbol{r}(\sigma) \tag{2.2}$$

and notice that

$$\ddot{\boldsymbol{q}}_i(\tau) = \Omega_i^2(\boldsymbol{r}(\tau) - \boldsymbol{q}_i(\tau)). \tag{2.3}$$

The equation of motion now takes the form

$$m\ddot{\boldsymbol{r}}(\tau) + \sum_{i=1}^{n} \frac{\kappa_i}{\Omega_i^2}\ddot{\boldsymbol{q}}_i(\tau) = \boldsymbol{F}(\tau). \tag{2.4}$$

Integrating twice, we find that

$$m\boldsymbol{r}(\tau)+\sum_{i=1}^{n}\frac{\kappa_i}{\Omega_i^2}\boldsymbol{q}_i(\tau)=\int_0^\tau \mathrm{d}\sigma(\tau-\sigma)\boldsymbol{F}(\sigma)+\boldsymbol{A}\tau+\boldsymbol{B} \tag{2.5}$$

where, since $\boldsymbol{q}_i(t)=\boldsymbol{q}_i(0)$,

$$\boldsymbol{A}t=m(\boldsymbol{r}(t)-\boldsymbol{r}(0))-\int_0^t \mathrm{d}\sigma(t-\sigma)\boldsymbol{F}(\sigma) \tag{2.6}$$

and

$$\boldsymbol{B}=m\boldsymbol{r}(0)+\sum_{i=1}^{n}\frac{\kappa_i}{\Omega_i^2}\boldsymbol{q}_i(0). \tag{2.7}$$

Then, substituting for $\boldsymbol{r}(\tau)$ from equation (2.3), we have an equation of motion for $\boldsymbol{q}_i(\tau)$

$$\ddot{\boldsymbol{q}}_i(\tau)+\sum_{j=1}^{n}P_{ij}\boldsymbol{q}_j(\tau)=\frac{\Omega_i^2}{m}\left(\int_0^\tau \mathrm{d}\sigma(\tau-\sigma)\boldsymbol{F}(\sigma)+\boldsymbol{A}\tau+\boldsymbol{B}\right) \tag{2.8}$$

where

$$P_{ij}=\Omega_i^2\left(\delta_{ij}+\frac{\kappa_j}{m\Omega_j^2}\right). \tag{2.9}$$

The matrix P is not symmetric but, in spite of this, it is a simple exercise to show that all its eigenvalues are real. It may also be shown that they are positive and distinct (as long as the Ω_i are distinct). These qualities are apparent from the following form of the eigenvalue equation:

$$\sum_{j=1}^{n}\frac{\kappa_j}{\omega_i^2-\Omega_j^2}=m \tag{2.10}$$

where ω_i^2 denotes an eigenvalue. Without loss of generality the ω_i and Ω_i may be arranged according to

$$\Omega_1^2<\omega_1^2<\Omega_2^2<\omega_2^2<\ldots<\Omega_n^2<\omega_n^2. \tag{2.11}$$

Since its eigenvalues are distinct, the matrix P must have a diagonal similar matrix D given by

$$P=V^{-1}DV \tag{2.12}$$

with

$$D_{ij}=\omega_i^2\delta_{ij}. \tag{2.13}$$

The transformation matrix V can be shown to be given by

$$(V^{-1})_{ij}=\frac{\Omega_i^2}{\omega_j^2-\Omega_i^2}. \tag{2.14}$$

If we now define

$$\boldsymbol{u}_i(\tau)=\sum_{j=1}^{n}V_{ij}\boldsymbol{q}_j(\tau) \tag{2.15}$$

and

$$h_i=\sum_{j=1}^{n}V_{ij}\frac{\Omega_j^2}{m} \tag{2.16}$$

the following equation of motion for $\boldsymbol{u}_i(\tau)$ can be deduced from equation (2.8):

$$\ddot{\boldsymbol{u}}_i(\tau)+\omega_i^2\boldsymbol{u}_i(\tau)=h_i\left(\int_0^\tau \mathrm{d}\sigma\,(\tau-\sigma)\boldsymbol{F}(\sigma)+\boldsymbol{A}\tau+\boldsymbol{B}\right). \tag{2.17}$$

The general solution for $\boldsymbol{u}_i(\tau)$ is

$$\boldsymbol{u}_i(\tau)=\boldsymbol{a}_i\cos\omega_i\tau+\boldsymbol{b}_i\sin\omega_i\tau+\frac{h_i}{\omega_i^2}\left(\int_0^\tau \mathrm{d}\sigma\,(\tau-\sigma)\boldsymbol{F}(\sigma)+\boldsymbol{A}\tau+\boldsymbol{B}\right)$$
$$-\frac{h_i}{\omega_i^3}\int_0^\tau \mathrm{d}\sigma\sin\omega_i(\tau-\sigma)\boldsymbol{F}(\sigma) \tag{2.18}$$

where

$$\boldsymbol{a}_i=\boldsymbol{u}_i(0)-\frac{h_i}{\omega_i^2}\boldsymbol{B} \tag{2.19}$$

and, since $\boldsymbol{u}_i(t)=\boldsymbol{u}_i(0)$,

$$\boldsymbol{b}_i\sin\omega_i t=\boldsymbol{a}_i(1-\cos\omega_i t)+\frac{h_i}{\omega_i^3}\int_0^t \mathrm{d}\sigma\sin\omega_i(t-\sigma)\boldsymbol{F}(\sigma)$$
$$-\frac{h_i}{\omega_i^2}m(\boldsymbol{r}(t)-\boldsymbol{r}(0)) \tag{2.20}$$

after substituting for $\boldsymbol{A}$ from equation (2.6).

Next, using equation (2.14), we may invert equations (2.15) and (2.16) to give respectively

$$\boldsymbol{q}_i(\tau)=\sum_{j=1}^{n}\frac{\Omega_i^2}{\omega_j^2-\Omega_i^2}\boldsymbol{u}_j(\tau) \tag{2.21}$$

and

$$m\sum_{j=1}^{n}\frac{h_j}{\omega_j^2-\Omega_i^2}=1. \tag{2.22}$$

Equation (2.21) leads, with equation (2.10), to the useful identity

$$\sum_{i=1}^{n}\frac{\kappa_i}{\Omega_i^2}\boldsymbol{q}_i(\tau)=m\sum_{i=1}^{n}\boldsymbol{u}_i(\tau) \tag{2.23}$$

which enables us to rewrite equation (2.5) as

$$m\boldsymbol{r}(\tau)=\int_0^\tau \mathrm{d}\sigma\,(\tau-\sigma)\boldsymbol{F}(\sigma)+\boldsymbol{A}\tau+\boldsymbol{B}-m\sum_{i=1}^{n}\boldsymbol{u}_i(\tau). \tag{2.24}$$

Also, from equations (2.7) and (2.19), we see that $\boldsymbol{B}$ is given by

$$\left(1-m\sum_{i=1}^{n}\frac{h_i}{\omega_i^2}\right)\boldsymbol{B}=m\left(\boldsymbol{r}(0)+\sum_{i=1}^{n}\boldsymbol{a}_i\right). \tag{2.25}$$

We now seek explicit expressions for $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$. From equation (2.2), substituting for $\boldsymbol{r}(\tau)$ from equation (2.24) and then for $\boldsymbol{u}_i(\tau)$ from equation (2.18), we see that

$$m\boldsymbol{q}_i(0)=\tfrac{1}{2}\Omega_i\int_0^t \mathrm{d}\tau\frac{\cos\Omega_i(t/2-\tau)}{\sin\frac{1}{2}\Omega_i t}\left\{\left(1-m\sum_{j=1}^n\frac{h_j}{\omega_j^2}\right)\left(\int_0^\tau \mathrm{d}\sigma\,(\tau-\sigma)\boldsymbol{F}(\sigma)+\boldsymbol{A}\tau+\boldsymbol{B}\right)\right.$$
$$-m\sum_{j=1}^n(\boldsymbol{a}_j\cos\omega_j\tau+\boldsymbol{b}_j\sin\omega_j\tau)$$
$$\left.+m\sum_{j=1}^n\frac{h_j}{\omega_j^3}\int_0^\tau \mathrm{d}\sigma\sin\omega_j(\tau-\sigma)\boldsymbol{F}(\sigma)\right\}. \tag{2.26}$$

Also, from equation (2.21) after substituting for $\boldsymbol{u}_i(0)$ from equation (2.19),

$$m\boldsymbol{q}_i(0)=m\sum_{j=1}^n\frac{\Omega_i^2}{\omega_j^2-\Omega_i^2}\left(\boldsymbol{a}_j+\frac{h_j}{\omega_j^2}\boldsymbol{B}\right). \tag{2.27}$$

These two expressions may be compared after noticing that, using equation (2.22),

$$m\sum_{j=1}^n\frac{\Omega_i^2}{\omega_j^2-\Omega_i^2}\frac{h_j}{\omega_j^2}=1-m\sum_{j=1}^n\frac{h_j}{\omega_j^2}. \tag{2.28}$$

The result of this comparison is that

$$\sum_{j=1}^n\frac{\Omega_i^2}{\omega_j^2-\Omega_i^2}\boldsymbol{a}_j$$
$$=\tfrac{1}{2}\Omega_i\int_0^t \mathrm{d}\tau\frac{\cos\Omega_i(t/2-\tau)}{\sin\frac{1}{2}\Omega_i t}\sum_{j=1}^n\left\{\frac{\Omega_i^2}{\omega_j^2-\Omega_i^2}\frac{h_j}{\omega_j^2}\left(\int_0^\tau \mathrm{d}\sigma\,(\tau-\sigma)\boldsymbol{F}(\sigma)+\boldsymbol{A}\tau\right)\right.$$
$$\left.-\boldsymbol{a}_j\cos\omega_j\tau-\boldsymbol{b}_j\sin\omega_j\tau+\frac{h_j}{\omega_j^3}\int_0^\tau \mathrm{d}\sigma\sin\omega_j(\tau-\sigma)\boldsymbol{F}(\sigma)\right\}. \tag{2.29}$$

Now, performing the integrals over τ and using equation (2.6) for $\boldsymbol{A}$ and equation (2.20) for $\boldsymbol{b}_i$, it can be shown that

$$\sum_{j=1}^n\frac{\omega_j}{\omega_j^2-\Omega_i^2}\left(\boldsymbol{a}_j\sin\omega_j t+\boldsymbol{b}_j(1-\cos\omega_j t)-\frac{h_j}{\omega_j^3}\int_0^t \mathrm{d}\tau\,(1-\cos\omega_j(t-\tau))\boldsymbol{F}(\tau)\right)=0. \tag{2.30}$$

Since the driving force $\boldsymbol{F}$ and the parameters t, $\boldsymbol{r}(t)$ and $\boldsymbol{r}(0)$ are all arbitrary, this equation may be satisfied only by setting the expression in parentheses equal to zero. We then have that

$$\boldsymbol{a}_i\sin\omega_i t+\boldsymbol{b}_i(1-\cos\omega_i t)=\frac{h_i}{\omega_i^3}\int_0^t \mathrm{d}\tau\,(1-\cos\omega_i(t-\tau))\boldsymbol{F}(\tau). \tag{2.31}$$

Solving together with equation (2.20), the final expressions for $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are found to be

$$\boldsymbol{a}_i=\tfrac{1}{2}m(\boldsymbol{r}(t)-\boldsymbol{r}(0))\frac{h_i}{\omega_i^2}+\frac{1}{2}\frac{h_i}{\omega_i^3}\frac{\sin\omega_i t}{1-\cos\omega_i t}\int_0^t \mathrm{d}\tau\,(1-\cos\omega_i(t-\tau))\boldsymbol{F}(\tau)$$
$$-\frac{1}{2}\frac{h_i}{\omega_i^3}\int_0^t \mathrm{d}\tau\sin\omega_i(t-\tau)\boldsymbol{F}(\tau) \tag{2.32}$$

and

$$\boldsymbol{b}_i = -\tfrac{1}{2}m(\boldsymbol{r}(t)-\boldsymbol{r}(0))\frac{h_i}{\omega_i^2}\frac{1+\cos\omega_i t}{\sin\omega_i t}+\frac{1}{2}\frac{h_i}{\omega_i^3}\int_0^t \mathrm{d}\tau\,(1-\cos\omega_i(t-\tau))\boldsymbol{F}(\tau)$$
$$+\frac{1}{2}\frac{h_i}{\omega_i^3}\frac{1+\cos\omega_i t}{\sin\omega_i t}\int_0^t \mathrm{d}\tau\sin\omega_i(t-\tau)\boldsymbol{F}(\tau). \tag{2.33}$$

The solution of the classical equation of motion is now given. From equations (2.24) and (2.18) we see that

$$m\boldsymbol{r}(\tau) = \left(1-m\sum_{i=1}^{n}\frac{h_i}{\omega_i^2}\right)\left(\int_0^\tau \mathrm{d}\sigma\,(\tau-\sigma)\boldsymbol{F}(\sigma)+\boldsymbol{A}\tau+\boldsymbol{B}\right)$$
$$-m\sum_{i=1}^{n}(\boldsymbol{a}_i\cos\omega_i\tau+\boldsymbol{b}_i\sin\omega_i\tau)$$
$$+m\sum_{i=1}^{n}\frac{h_i}{\omega_i^3}\int_0^\tau \mathrm{d}\sigma\sin\omega_i(\tau-\sigma)\boldsymbol{F}(\sigma) \tag{2.34}$$

where $\boldsymbol{a}_i$, $\boldsymbol{b}_i$, $\boldsymbol{A}$ and $\boldsymbol{B}$ are given by equations (2.32), (2.33), (2.6) and (2.25) respectively.

The classical action S_{cl} is given by substitution of the classical solution into equation (1.1). Firstly, however, it is useful to show, from equations (1.1) and (2.1), that

$$S_{\mathrm{cl}} = \tfrac{1}{2}m(\dot{\boldsymbol{r}}(t)\cdot\boldsymbol{r}(t)-\boldsymbol{r}(0)\cdot\dot{\boldsymbol{r}}(0))+\tfrac{1}{2}\int_0^t \mathrm{d}\tau\,\boldsymbol{F}(\tau)\cdot\boldsymbol{r}(\tau) \tag{2.35}$$

with $\boldsymbol{r}(\tau)$ given by equation (2.34). The velocity $\dot{\boldsymbol{r}}(\tau)$ is given, after differentiating equation (2.34), by

$$m\dot{\boldsymbol{r}}(\tau) = \left(1-m\sum_{i=1}^{n}\frac{h_i}{\omega_i^2}\right)\left(\int_0^\tau \mathrm{d}\sigma\,\boldsymbol{F}(\sigma)+\boldsymbol{A}\right)+m\sum_{i=1}^{n}\omega_i(\boldsymbol{a}_i\sin\omega_i\tau-\boldsymbol{b}_i\cos\omega_i\tau)$$
$$+m\sum_{i=1}^{n}\frac{h_i}{\omega_i^2}\int_0^\tau \mathrm{d}\sigma\cos\omega_i(\tau-\sigma)\boldsymbol{F}(\sigma). \tag{2.36}$$

A final expression for S_{cl} is obtained by substitution into equation (2.35). The manipulations are straightforward but somewhat lengthy. We find that

$$S_{\mathrm{cl}} = \left\{\tfrac{1}{4}m^2\sum_{i=1}^{n}\frac{h_i}{\omega_i}\cot\tfrac{1}{2}\omega_i t+\frac{m}{2t}\left(1-m\sum_{i=1}^{n}\frac{h_i}{\omega_i^2}\right)\right\}(\boldsymbol{r}(t)-\boldsymbol{r}(0))^2$$
$$+\tfrac{1}{2}(\boldsymbol{r}(t)+\boldsymbol{r}(0))\int_0^t \mathrm{d}\tau\,\boldsymbol{F}(\tau)+(\boldsymbol{r}(t)-\boldsymbol{r}(0))\int_0^t \mathrm{d}\tau\,\boldsymbol{F}(\tau)$$
$$\times\left\{\left(1-m\sum_{i=1}^{n}\frac{h_i}{\omega_i^2}\right)\left(\frac{\tau}{t}-\frac{1}{2}\right)+m\sum_{i=1}^{n}\frac{h_i}{\omega_i^2}\left(\frac{\sin\omega_i\tau-\sin\omega_i(t-\tau)}{2\sin\omega_i t}\right)\right\}$$
$$-\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}\sigma\,\boldsymbol{F}(\tau)\cdot\boldsymbol{F}(\sigma)\left\{\frac{1}{mt}\left(1-m\sum_{i=1}^{n}\frac{h_i}{\omega_i^2}\right)(t-\tau)\sigma+\sum_{i=1}^{n}\frac{h_i}{\omega_i^3\sin\omega_i t}\right.$$
$$\left.\times(\sin\omega_i(t-\tau)\sin\omega_i\sigma-4\sin\tfrac{1}{2}\omega_i\tau\sin\tfrac{1}{2}\omega_i(t-\tau)\sin\tfrac{1}{2}\omega_i\sigma\sin\tfrac{1}{2}\omega_i(t-\sigma))\right\}. \tag{2.37}$$

For the case of one oscillator ($n=1$) it is a simple exercise to show that this reduces to the same expression as given by Sa-yakanit [3].

We now have a complete solution of the classical problem concerning a particle of mass m interacting with n other particles of mass $M_i = \kappa_i/\Omega_i^2$ and also being under the influence of an arbitrary driving force. This system will be translationally invariant only if

$$\int_0^t \mathrm{d}\tau\, \boldsymbol{F}(\tau) = 0. \tag{2.38}$$

3. The prefactor

The prefactor $G(t)$ of the propagator is defined in equation (1.4). An explicit expression for this function may be given from the classical solution by introducing a generating functional.

Firstly, from equation (1.4), we see that

$$-\mathrm{i}\hbar\kappa_i \frac{\partial}{\partial\kappa_i} \ln G(t) = \left\langle \kappa_i \frac{\partial S_{F=0}}{\partial \kappa_i} \right\rangle \tag{3.1}$$

where the average $\langle x\rangle$, for any quantity x, is given by

$$\langle x\rangle = \frac{1}{G(t)} \int_0^0 \mathrm{D}[\boldsymbol{r}(\tau)]x \exp\left[\frac{\mathrm{i}}{\hbar} S_{F=0}\right]. \tag{3.2}$$

Then, substituting for the action from equation (1.1),

$$-\mathrm{i}\hbar\kappa_i \frac{\partial}{\partial\kappa_i} \ln G(t) = -\tfrac{1}{8}\kappa_i\Omega_i \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\sigma \frac{\cos \Omega_i(t/2 - |\tau-\sigma|)}{\sin \frac{1}{2}\Omega_i t} \langle(\boldsymbol{r}(\tau) - \boldsymbol{r}(\sigma))^2\rangle. \tag{3.3}$$

Secondly, we introduce a generating functional

$$Z[\boldsymbol{F}] = \left\langle \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}\tau'\, \boldsymbol{F}(\tau')\cdot \boldsymbol{r}(\tau')\right]\right\rangle. \tag{3.4}$$

If the function $\boldsymbol{F}$ is chosen to be given by

$$\boldsymbol{F}(\tau') = \hbar\boldsymbol{k}(\delta(\tau-\tau') - \delta(\sigma-\tau')) \tag{3.5}$$

then

$$Z[\boldsymbol{F}] = \langle \exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}(\tau) - \boldsymbol{r}(\sigma))]\rangle. \tag{3.6}$$

The generating functional may be expressed explicitly by substituting for $\boldsymbol{F}$ from equation (3.5) into equation (2.37) for the classical action and setting $\boldsymbol{r}(t) = \boldsymbol{r}(0) = 0$. The result is that

$$\langle \exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}(\tau) - \boldsymbol{r}(\sigma))]\rangle = \exp[-\tfrac{1}{2}\mathrm{i}\hbar k^2 R(|\tau-\sigma|)] \tag{3.7}$$

with

$$R(x) = 2\sum_{i=1}^{n} \frac{h_i}{\omega_i^3} \frac{\sin\frac{1}{2}\omega_i x \sin\frac{1}{2}\omega_i(t-x)}{\sin\frac{1}{2}\omega_i t} + \frac{1}{mt}\left(1 - m\sum_{i=1}^{n}\frac{h_i}{\omega_i^2}\right)x(t-x). \tag{3.8}$$

Next, expanding equation (3.7) in powers of k^2 for each space dimension, we have that

$$\langle(\boldsymbol{r}(\tau) - \boldsymbol{r}(\sigma))^2\rangle = \mathrm{i}\hbar\; \mathrm{d}R(|\tau-\sigma|) \tag{3.9}$$

where d is the number of space dimensions. Now, substituting into equation (3.3), it can be shown that

$$\frac{\partial}{\partial \kappa_i} \ln G(t) = \tfrac{1}{4} \mathrm{d}\Omega_i \int_0^t \mathrm{d}x\,(t-x) \frac{\cos \Omega_i(t/2-x)}{\sin \frac{1}{2}\Omega_i t} R(x). \tag{3.10}$$

Performing the integrals and making use of equation (2.28) then leaves us with

$$\frac{\partial}{\partial \kappa_i} \ln G(t) = \tfrac{1}{2} \mathrm{d} \sum_{j=1}^{n} \frac{h_j}{\omega_j} \frac{1}{\omega_j^2 - \Omega_i^2} \left(\frac{1}{\omega_j} - \tfrac{1}{2} t \cot \tfrac{1}{2}\omega_j t \right). \tag{3.11}$$

In order to proceed further we first need to invert equation (2.10) to give an expression for κ_i in terms of the ω_i and Ω_i. The result, after some algebraic manipulations, is

$$\kappa_i = m(\omega_i^2 - \Omega_i^2) \prod_{j=1}^{n}{}' \frac{\omega_j^2 - \Omega_i^2}{\Omega_j^2 - \Omega_i^2} \tag{3.12}$$

where the prime on the product indicates that the $j = i$ factor is excluded. Similarly inverting equation (2.22) we find that

$$h_i = \frac{1}{m} (\omega_i^2 - \Omega_i^2) \prod_{j=1}^{n}{}' \frac{\Omega_j^2 - \omega_i^2}{\omega_j^2 - \omega_i^2}. \tag{3.13}$$

From these two expressions it can be shown that

$$\frac{\partial \omega_j}{\partial \kappa_i} = \frac{1}{2} \frac{h_j}{\omega_j} \frac{1}{\omega_j^2 - \Omega_i^2} \tag{3.14}$$

which, with equation (3.11), means that

$$\frac{\partial}{\partial \omega_j} \ln G(t) = d \left(\frac{1}{\omega_j} - \tfrac{1}{2}\, t \cot \tfrac{1}{2}\, \omega_j t \right). \tag{3.15}$$

Integrating this equation and comparing with the free-particle limit ($\kappa_i = 0$, $\omega_i = \Omega_i$) finally gives

$$G(t) = \left(\frac{m}{2\pi \mathrm{i} \hbar t} \right)^{d/2} \prod_{i=1}^{n} \left(\frac{\omega_i}{\Omega_i} \frac{\sin \frac{1}{2}\Omega_i t}{\sin \frac{1}{2}\omega_i t} \right)^d. \tag{3.16}$$

In this expression, the first factor is the prefactor for the free-particle propagator [5]. For the case of one oscillator ($n = 1$) this expression agrees with that of [3].

We now have a complete expression for the propagator corresponding to the action of equation (1.1). It is given by equations (1.3), (2.37) and (3.16).

4. Concluding remarks

We have given a complete expression for the propagator corresponding to the motion of a series of any number of non-local harmonic oscillators with an arbitrary driving force present. Our expression is complete in the sense that the parameters t, $\boldsymbol{r}(t)$ and $\boldsymbol{r}(0)$ are all kept completely general.

Concerning the use of this propagator in variational calculations, the original expression for the action contained $2n$ variational parameters (κ_i and Ω_i for $1 \leqslant i \leqslant n$). In contrast, the final expression for the propagator does not depend explicitly on the

κ_i. However, we may equally well choose our $2n$ variational parameters to be the ω_i and Ω_i with h_i given by equation (3.13). The essential point here is that it is not necessary to perform any matrix diagonalizations in order to estimate physical quantities by variational techniques using the trial action of equation (1.1).

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